Optimal Replication of Futures and Options on Chinese Market Restrictions

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2. Abstract

This paper develops a future and option pricing model and applies it to the Chinese market. We incorporate a rich set of Chinese market restrictions, including transaction costs, constraints on shorting stocks, the maximum stock price change in a given day, and the T+1 rule (refers to the policy that asserts that the earliest allowed stock selling date is one day after purchase date). We use linear programming to solve the model, and test the model using data.

3. Introduction

This paper investigates the problem of replicating a given payoff in an imperfect market. By replicating a given payoff, the investor can price futures as well as put and call options. The most revolutionary contribution to option pricing is from Black and Scholes (1973) [1] and Merton (1973) [2]. In their model, they price options, relying on the arbitrage argument, by replicating the payoff using a portfolio consisting of the underlying stock and a risk-free bond. Due to the nature of the model, an investor must continuously trade as time passes for the resulting payoff to be exactly replicated. With the introduction of transaction costs, this approach would be unreasonably expensive due to the high frequency of trading. A remedy for this problem is first introduced by Gilster and Lee (1984) [3] and Leland (1985) [4] by modifying the Black-Scholes and Merton (BSM) option pricing formula to incorporate transaction cost. However, these models are not flexible enough to take into account more restricting constraints.

An alternative, and more convenient, approach is to use a discrete-time framework. By using the binomial model developed by J. C. Cox et al. (1979) [5], one can price options based on the final payoff. In addition to being simple and intuitive, this approach converges to BSM option pricing formula as the time interval between stock movements becomes increasingly small. Merton (1990) [6] and Boyle and Vorst (1992) [7] use this approach to evaluate the impact of transaction costs on option pricing by exactly matching the payoff. Bensaid et al. (1992) [8] shows that in the presence of transaction costs, a cheaper replicating portfolio can be found by "super-replicating" the required payoff instead of matching it exactly; these results are extended by Perrakis and Lefoll (1997) [9].

Other authors have explored different settings to assess the effect of transaction costs on option pricing. Hodges and Neuberger (1989) [10] assume a particular utility function for an individual and based on that try to maximize the expected utility realized. This model is further developed by Davis et al. (1993) [11], and is used by Whalley and Willmott (1997) [12], Constantinides and Zariphopoulou (1999) [13]. Since this approach depends on an individual's preference and probability beliefs, it becomes a setback when dealing with institutions and market makers.

In this paper, we use the methodology developed by Edirisinghe et al. (1993) [14] which adopts an optimization criterion by minimizing the initial cost of replicating a desired terminal payoff. This approach utilizes the “super-replicating” procedure to obtain a narrow valuation bound. Unlike the utility function approach, the optimal solution obtained here is independent of the investor’s preferences and probability beliefs.

With the analysis presented here, the current literature based on finding the optimal replicating portfolio is extended in several ways. In addition to the variable transaction costs and the requirement to dominate the terminal payoff, which are presented in Bensaid et al. (1992) [8] and Edirisinghe et al. (1993) [14], our formulation adds a variety of additional practical constraints. Due to the risky nature of shorting shocks, many markets do not allow such action. Hence, we modify the existing model in a way that would prevent shorting stocks. Moreover, the (T+1) rule is incorporated in the model. The (T+1) rule states that a stock purchased in a given trading period cannot be sold until the next trading period. Finally, the maximum stock change in a given day is 10% (increase or decrease) as compared to the stock price at the beginning of the trading period. The latter two constraints are specific to the Chinese market.

A limitation in the binomial tree developed here is that there is no rule for determining the number of steps to use in the model in order to obtain reasonable results. This is critical since as the number of steps is increased, the option valuation spread between the upper and lower bounds increases as well. The main reason for this increase in spread is because transaction costs tends to consume the initial wealth as the number of trading periods is increased. Moreover, as conjectured by Davis and Clark (1994) [15], later proved by Soner et al. (1995) [16], extending this model into a continuous-time framework (by increasing the number of steps indefinitely) would result in the trivial strategy of purchasing one underlying stock and keeping it till maturity as the optimal replicating portfolio; this strategy would avoid transaction costs and, at the same time, super replicate the payoff. However, Henrotte (1993) [17] and Flesaker and Hughston (1994) [18] propose a solution to prevent this trivial situation by adding an assumption on the proportion of transaction cost. Instead of keeping the proportion transaction cost fixed, they replace it by a parameter that declines in proportion to the square root of the number of binomial steps. Nevertheless, even without this assumption, the optimal replication obtained from the discrete-time setting (with a finite number of steps) can help us assess the impact of various restrictions on the valuation and provides a benchmark for the output of different methods. Moreover, most extensions (*e.g.* daily stock price limit, T+1 rule) added in this paper can be applied without any transaction cost. Furthermore, when the transaction cost is low (1% or lower), the increase in option price with each step becomes less significant. Knowing that the stock market's transaction costs are to the order of 0.01%-0.1% reinforces the applicability of the model.

This article is organized as follows. The basic model used to apply the restrictions is explained in Section 4. Section 5.1 introduces restrictions on shorting stocks and shows results from applying this restriction. The daily stock price limit is introduced in Section 5.2 along with some approximations to make it practical. Finally, Section 5.3 incorporates the T+1 rule and illustrates its effect on the option's price with a numerical example.

4. The Basic Model

The discrete time model developed by J. C. Cox et al. (1979) [5] is used here. The notation used here is similar to the one used by Edirisinghe et al. (1993) [14]. All possible paths for the stock price are represented by an event tree. The number of steps on the event tree is N, T refers the number of time periods and P is the number of steps per period. Time takes discrete values in the set {0, 1, 2, …, N}. All possible events at each time period is contained by a family, $\mathcal{F}_n$, for n=1, 2, …, N. For any event, $j \in \mathcal{F}_n$, there is a unique predecessor, $i \in \mathcal{F}_{n-1}$, such that $j \subset i$. Moreover, for each event $j \in \mathcal{F}_n$ there are two successive elements $j_u, j_d \in \mathcal{F}_{n+1}$. The set of all possible events at $\mathcal{F}_N$ are denoted by Ω, which represents the list of all possible outcomes. An example of an event tree is shown in Figure 1 with P=3 and T=1. In this event tree, $\mathcal{F}_1$ is represented by the set $\{S_u, S_d\}$, $\mathcal{F}_2$ is represented by $\{S_{uu}, S_{ud}, S_{du}, S_{dd}\}$, and $\mathcal{F}_3 = \Omega$ is represented by the set $\{S_{uuu}, S_{uud}, S_{udu}, \ldots, S_{ddd}\}$.

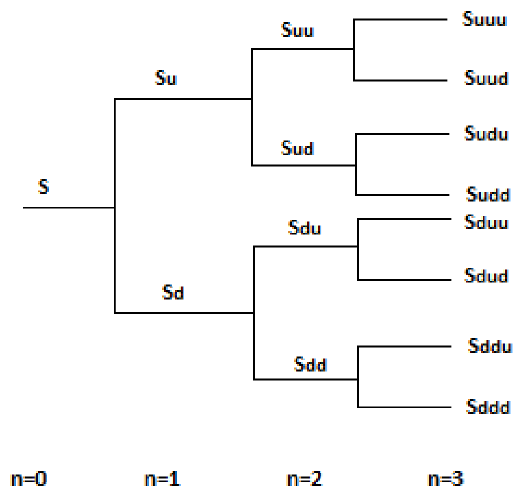

Figure 1: Binomial Tree with P=3

In order to price the option, it is necessary to replicate the payoff of an option at the terminal step (*N*). The replicating portfolio used consists of a position in the underlying stock and pure discount riskless bond maturing at *N*. Let $S(n, j)$ denote the stock price at step n in event j, where $j \in \mathcal{F}_n$. It is assumed that the stock price is generated using the binomial tree shown in Figure 1. More specifically, $S(n+1, j_u) = uS(n, j)$ and $S(n+1, j_d) = dS(n, j)$ where $d < 1 < u$. For simplicity, it is assumed that there is no interest rate. This means that if the price of a pure discount bond is $B(n, N)$ at n, then $B(n, N) = 1 \; \forall \, n$. Moreover, assume that $d = 1/u$ to make sure that the tree recombines. However, the formulation can easily incorporate the cases where the interest rate is positive and $d \neq 1/u$.

The objective of the problem is to minimize the initial cost of the portfolio that is capable of replicating the option's payoff. The transaction cost is modeled to be a proportion (θ) to the value of stocks being traded. Hence, the transaction cost in buying or selling k stocks at step n and event j is $k\theta S(n, j)$. Let $\alpha(n, j)$ and $\beta(n, j)$ denote the stock and bond position, respectively, at step n in event j, where $j \in \mathcal{F}_n$. As Edirisinghe et al. (1993) [14] shows, this problem can be modeled as a Linear Program (LP). Let $x(n, j)$ and $y(n, j)$ be defined as the number of additional shares bought and additional shares sold, respectively, at step n and event j. Also, let $C(N, j)$ denote the desired payoff at the terminal step. At n=0, the number of shares, bond position and stock price are denoted by $\alpha(0)$, $\beta(0)$ and $S(0)$ respectively. Then the problem is represented by (denote the LP by **L1**)

(L1) $$min_{\alpha,\beta}\ \alpha(0)S(0)+\beta(0) \tag{1}$$

subject to trading constraints,

$$x(n,j)(1+\theta)S(n,j)-y(n,j)(1-\theta)S(n,j)+\beta(n,j)-\beta(n-1,i)\le 0 \tag{2}$$

$$\alpha(n,j)-\alpha(n-1,i)=x(n,j)-y(n,j) \tag{3}$$

$$x(n,j),y(n,j)\ge 0 \tag{4}$$

$$\forall j\in\mathcal{F}_n, j\subset i, i\in\mathcal{F}_{n-1}, n=1,2,\dots,N-1$$

And subject to the super-replicating constraints,

$$\alpha(N-1,i)S(N,j)+\beta(N-1,i)\ge C(N,j),\forall j\in\mathcal{F}_N, j\subset i, i\in\mathcal{F}_{N-1} \tag{5}$$

Constraint (2) imposes the restriction that any trade performed at time *t* must be executable without the need for external funds; hence the only cost (*c*) incurred is the cost used to purchase the initial position $c=\alpha(0)S(0)+\beta(0)$. Moreover, constraint (5) ensures that the replicating portfolio has a value at least as large as the option's payoff; hence, it is referred to as a super-replicating portfolio. This formulation is path dependent which means that the trading constraints need to be imposed path-by-path. As a result, the number of variables and constraints increase exponentially with N.[1] To reduce the dimension of the problem, an approximate LP that is path independent is formulated by Edirisinghe et al. (1993) [14]. This model is obtained by adding the following restrictions to the original problem

(L2) $$\alpha(n,i)=\alpha(n,j)\ \forall i,j\in\mathcal{F}_n\ when\ S(n,i)=S(n,j), n=2,\dots,N-1 \tag{6}$$

$$\beta(n,i)=\beta(n,j)\ \forall i,j\in\mathcal{F}_n\ when\ S(n,i)=S(n,j), n=2,\dots,N-1 \tag{7}$$

By adding these restrictions, the number of variables and constraints grows quadratically instead.[2] The previous models (**L1** and **L2**) assume no transaction cost is incurred when obtaining the initial position. This assumption degrades the pricing accuracy, especially when shorting cost is introduced. If we let *w* denote the initial wealth required, then the following modified version of **L1** can be used to discard this assumption

(L3) $$min_{\alpha,\beta}\ w \tag{8}$$

[1] Exact LP: the number of variables and constraints are $2^{2+N}-6$ and $5(2^N)-8$ respectively.
[2] Approximate LP: the number of variables and constraints are $3N^2-N$ and $4N^2-2N$ respectively.

Subject to trading constraints,

$$x(0)(1+\theta)S(0) - y(0)(1-\theta)S(0) + \beta(0) - w \le 0 \tag{9}$$

$$x(n,j)(1+\theta)S(n,j) - y(n,j)(1-\theta)S(n,j) + \beta(n,j) - \beta(n-1,i) \le 0 \tag{10}$$

$$\alpha(0) = x(0) - y(0) \tag{11}$$

$$\alpha(n,j) - \alpha(n-1,i) = x(n,j) - y(n,j) \tag{12}$$

$$x(n,j), y(n,j) \ge 0 \tag{13}$$

$$\forall j \in \mathcal{F}_n, j \subset i, i \in \mathcal{F}_{n-1}, n = 1,2,\dots,N-1$$

And subject to the super-replicating constraints,

$$\alpha(N-1,i)S(N,j) + \beta(N-1,i) \ge C(N,j), \forall j \in \mathcal{F}_N, j \subset i, i \in \mathcal{F}_{N-1} \tag{14}$$

Moreover, to reduce the computational requirements of the model, restrictions (6) and (7) can be applied to **L3**, resulting in an approximate LP which will be referred to as **L4**. Table 1 displays some results from applying **L3** and **L4** on a put option with a one month maturity. In the table, K represents the strike price, σ is the standard deviation per period, and Δ is the step size. It is clear from the table that, in addition to the lower dimensionality, the approximate LP (**L4**) produces accurate results when compared to the exact path dependent LP (**L4**). Discrepancies between the two are observed when the transaction cost is large (more than 1%). Practically, however, the transaction cost is almost always less than 1%; hence the approximate LP (**L4**) can be used with a great level of confidence. Table 2 shows similar results for a call option.

Table 1: Cost of Replicating a Long Put Option (S=100, T=1, P=8, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$)

Transaction cost (θ)	K=95		K=100		K=105	
	σ=0.05	σ=0.1	σ=0.05	σ=0.1	σ=0.05	σ=0.1
0%	0.41[a]	1.95	1.93	3.87	5.47	7.14
	0.41[b]	1.95	1.93	3.87	5.47	7.14
0.01%	0.42	1.96	1.95	3.88	5.48	7.16
	0.42	1.96	1.95	3.88	5.48	7.16
0.1%	0.48	2.07	2.09	4.02	5.61	7.30
	0.48	2.07	2.09	4.02	5.61	7.30
1%	1.22	3.09	3.34	5.33	6.85	8.64
	1.22	3.09	3.34	5.33	6.85	8.64
5%	4.49	7.17	7.59	10.06	11.27	13.56
	4.59	7.28	7.71	10.16	11.31	13.66

[a] Evaluated using exact LP **(L3)**
[b] Evaluated using approximate LP **(L4)**

Table 2: Cost of Replicating a Long Call Option (S=100, T=1, P=8, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$)

Transaction cost (θ)	K=95		K=100		K=105	
	σ=0.05	σ=0.1	σ=0.05	σ=0.1	σ=0.05	σ=0.1
0%	5.41[a]	6.95	1.93	3.87	0.47	2.14
	5.41[b]	6.95	1.93	3.87	0.47	2.14
0.01%	5.42	6.97	1.95	3.88	0.47	2.16
	5.42	6.97	1.95	3.88	0.47	2.16
0.1%	5.55	7.11	2.09	4.03	0.55	2.27
	5.55	7.11	2.09	4.03	0.55	2.27
1%	6.77	8.46	3.37	5.38	1.37	3.38
	6.77	8.46	3.37	5.38	1.37	3.38
5%	11.25	13.54	7.80	10.41	4.98	7.93
	11.27	13.64	7.89	10.53	5.11	8.05

[a] Evaluated using exact LP **(L3)**
[b] Evaluated using approximate LP **(L4)**

5. Extensions

This section introduces a variety of restrictions that are applicable to a wide variety of stock markets. These restrictions can be applied individually or they can be combined together depending on the application. The following sections not only provide basic formulation, but numerical results are also evaluated to get a sense of the impact of these restrictions on the price of the option.

5.1. Short Selling Restrictions

Shorting stocks is not allowed in many stock markets. In order to take this into account, we can add the following as an additional constraint to **L4**

(L5) $$\alpha(n,j) \geq 0 \; \forall j \in \mathcal{F}_n, n = 0,1,2,\dots,N-1 \quad (15)$$

This restriction reveals some interesting results when applied to pricing call and put options. First, the pricing of the call option is the same whether or not we include constraint (15); this is attributed to the fact that when replicating a call's payoff, the optimal replicating portfolio never maintains a negative stock position (in *any* node). Second, the price of the put option is drastically increased when constraint (7) is applied. Again, observing the replicating portfolio reveals that in order to match the put's payoff, it always maintains a negative stock position. Naturally, when constraint (7) is added, it would prevent the stock position from going negative and hence it will remain zero at all nodes. As a consequence, the replicating portfolio only holds a bond position; the bond position held should be at least as large as the highest payoff required (K). Since the portfolio is not revised in each step, the transaction cost does not affect the put option's price. The effect of this constraint on the put option pricing is shown in the last row of Table 3. In the table, the pricing in the case of disallowed shorting does not converge to K due to the reduced number of steps used; increasing the number of steps would cause the pricing to converge to K.

Instead of disallowing short selling stocks, some markets allow shorting stock but require extra fees in the form of insurance. We can incorporate these extra fees as a higher transaction cost required for shorting. Similar to the case where shorting is disallowed, this will only affect the price of put options. Moreover, since the put option's replicating portfolio never maintains a positive stock position, any stock sold can be considered as a stock shorted. Due to this observation, costly shorting can be modeled by editing the trading constraints in **L4** to

(L6)
$$x(0)(1+\theta)S(0) - y(0)(1-\delta)S(0) + \beta(0) - w \leq 0 \quad (16)$$
$$x(n,j)(1+\theta)S(n,j) - y(n,j)(1-\delta)S(n,j) + \beta(n,j) - \beta(n-1,i) \leq 0 \quad (17)$$

where δ is the cost of shorting and θ is the transaction cost as a proportion of the traded stock value. As the shorting transaction cost increases, the price of the put option increases as shown in Table 3. However, there is a point after which the price of the put option remains the same regardless of how much the sorting transaction cost is increased afterwards. At this point it stops becoming economical to short stocks and it would be best to simply maintain a bond position that would cover the highest payoff. Clearly, increasing the cost of shorting significantly would cause the price to converge to the price of a put with shorting disallowed.

Table 3: Price of Put Option with Costly Shorting (S=100, K=100, T=1, P=20, Δ=T/N, σ=0.1, $u = e^{\sigma\sqrt{\Delta}}$, θ=0%)

Shorting Transaction Cost (δ)	K=95	K=100	K=105
0%	1.88	3.94	7.04
0.01%	1.89	3.95	7.06
0.1%	1.98	4.07	7.19
0.5%	2.39	4.60	7.75
1%	2.90	5.23	8.43
2%	3.89	6.42	9.71
5%	6.71	9.67	13.16
10%	11.10	14.50	18.26
20%	19.46	23.45	27.64
40%	31.06	36.06	41.06
Shorting Not Allowed	31.06	36.06	41.06

To further determine the effect of the shorting cost (δ) as opposed to the transaction cost (θ) on put option pricing, Figures 2 and 3 are presented. These figures display the effect of the shorting and repayment cost on the implied volatility. The implied volatility is another method to assess the price of the option. One advantage of using implied volatility instead of price is that it can be readily compared with the volatility of the underlying stock. For both in-money and out-of-money put options the implied volatility is more sensitive to the shorting cost. This can be observed by comparing the slope of the line when the repayment cost is zero and the slope when the shorting cost is zero. Moreover, the implied volatility seems to increase linearly when the shorting and repayment cost are increased. This also means that there is no interaction effect between the two. Finally, there does not seem to be any difference between the shape of the implied volatility graph of the both cases: in-money and out-of-money. The only difference between the two is the implied volatility range, where the in-money put has a higher range.

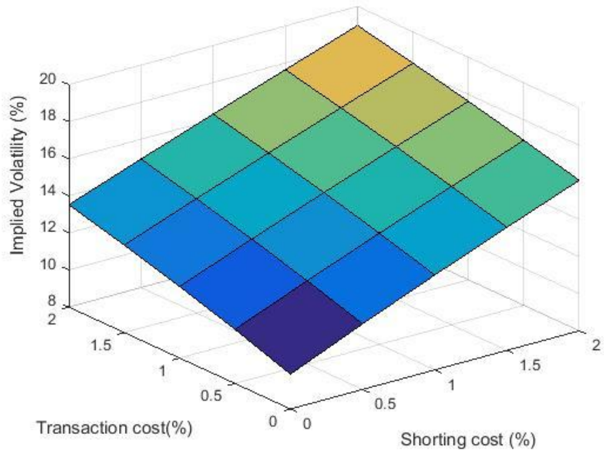

Figure 2: Implied volatility for at the money put option (S=100,K=100, σ=0.1, T=1, P=20)

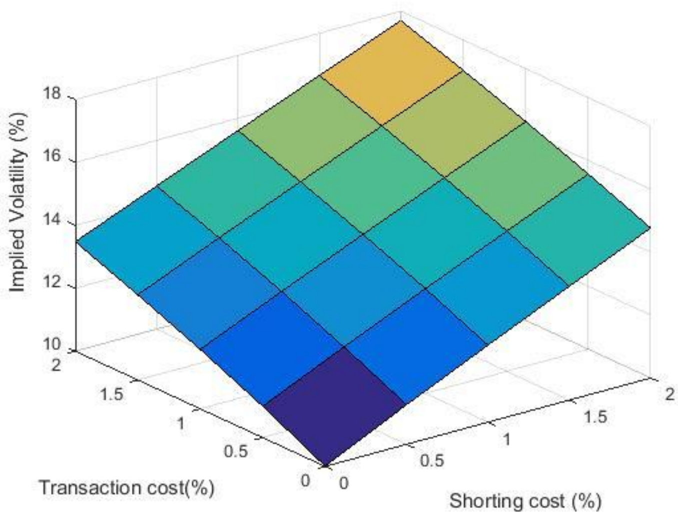

Figure 3: Implied Volatility for out-of-money Put Option (S=100,K=85, σ=0.1, T=1, P=20)

5.2. Stock Price Limit

Some financial markets impose a daily limit on the amount on which the stock price can increase (or decrease). The main rational for imposing stock price limit is to prevent surge changes in stock price, especially in emerging companies, which keeps the market stable. Moreover, when the stock price is at the upper limit, an investor cannot buy shares. And when the stock price is at the lower limit, the investor cannot sell shares. This stock price limit would no doubt have an effect on the price of an option. In order to incorporate this restriction into the model, the number of days (D) and the number of periods in a day (P) must be specified. Using

D and P, the total number of steps can be calculated with $N = D * P$. To maintain some consistency with the previous results, T is used to denote the time to maturity in months. Since there are 21 trading days in a month, T can be computed with $T = D/21$. Define $S^{tk}(n, j)$ as the stock price at period *n* in day *t* undergoing event *j*, where *n=1,2,…,P*. Furthermore, *k* represents the specific node used to start the stock price for the next day. The range of *k* in a specific day depends on the number of end nodes in the previous day. Unless *t=D*, each end node in a given day must start a new tree in the next day. Define $S^{tk}(0)$ as the initial stock price at day *t* stating at node *k*. If the upper and lower daily limits are given by *U* and *L*, respectively, where *L<1<U* then the following restrictions can be added:

(L7)

$$S^{tk}(n,j) = S^{tk}(0) * U \; when \; S^{tk}(n,j) > S^{tk}(0) * U \quad (18)$$

$$S^{tk}(n,j) = S^{tk}(0) * L \; when \; S^{tk}(n,j) < S^{tk}(0) * L \quad (19)$$

$$t = 1, 2, \ldots, D \quad n = 1, 2, \ldots, P \quad \forall j \in \mathcal{F}_{(t-1)*P+n}$$

$$x(n,j) = 0 \; when \; S^{tk}(n,j) = S^{tk}(0) * U \quad (20)$$

$$y(n,j) = 0 \; when \; S^{tk}(n,j) = S^{tk}(0) * L \quad (21)$$

$$t = 1, 2, \ldots, D \quad n = 1, 2, \ldots, P \quad \forall j \in \mathcal{F}_{(t-1)*P+n}$$

Clearly, the previous restrictions are path dependent. The results from **L7** are plotted in Figure 4. Looking closely at the figure, the price of the put option seems to follow a specific trend. At some points, the option's price is below the no limit case and at others it is above. This behavior can be explained by two competing forces. On one hand, the stock price limit reduces volatility which in turn reduces the option's price. On the other hand, when the stock price is at the upper (lower) limit, buying (selling) is restricted. When buying (or selling) is restricted, it becomes more difficult to replicate the option's payoff, driving up the price. Moreover, at some points, there is an obvious price drop; this price drop occurs the moment the stock price limit surpasses a step increase in the basic model ($U = u^p$ where p is an integer). When the stock price limit surpasses a step increase, there is more room for the stock to change before reaching the upper (or lower) limit. This reduces the buying (or selling) restriction causing the price drop. The effect of the two forces decreases after each step till it diminishes when the limits become out of bound and the volatility return to the normal case. Increasing the number of days under study would produce more practical results. Due to the exponential growth from the exact model, increasing the number of days is computational infeasible in the current model.[3] To run the model for an extended period of time, an approximate model is developed in the next section.

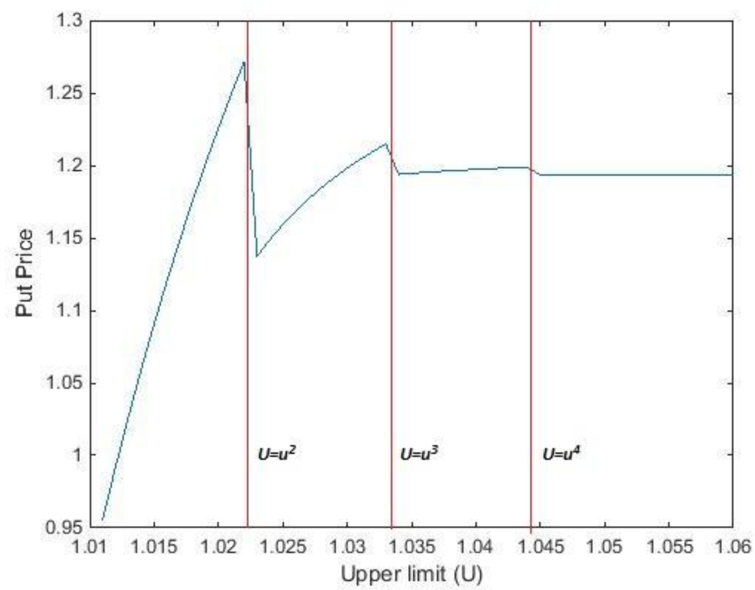

Figure 4: Price of Put Option as the limits are increased (S=100, K=100, L=1/U, σ=0.1, D=2, P=4, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$, θ=0%)

5.2.1. Approximating the Stock Price Limit

The exact LP retains the whole path of the stock at every single step. Instead of retaining the whole path, the approximate LP can be used to give an accurate solution. In addition to using the approximate LP discussed in the basic model, a state variable is added at the beginning of each day to keep track of the starting stock price. To apply the state variable, define $\alpha^{tk}(n,j)$ as the stock position at step n in day t undergoing event j. Moreover, k is specified to discern the stock price state at the beginning of each day. The range of k depends on the number of distinct stock prices at the end of each day. The expression $\alpha^{(t-1)k}(P,j) = \alpha^{tm}(0,j)$ where $, m \subset k$ provides a link between the end of a day and the beginning of the next. In addition, to apply the approximate LP, set $\alpha^{tk}(n,i) = \alpha^{tk}(n,j)$ $\forall i,j \in \mathcal{F}_{(t-1)*P+n}$ when $S^{tk}(n,i) = S^{tk}(n,j)$ $\forall\, k, t = 1, \ldots, D\ n = 1, \ldots, P$. Also, define $\beta^{tk}(n,j)$ in a similar way and set $\beta^{tk}(n,i) = \beta^{tk}(n,j)$ when $S^{tk}(n,i) = S^{tk}(n,j)\ \forall i,j \in \mathcal{F}_{(t-1)*P+n}$ $\forall\, k, t = 1, \ldots, D\ \ n = 1, \ldots, P$. Finally, $x^{tk}(n,j)$ and $y^{tk}(n,j)$ are defined in a similar way.

[3] The maximum number of periods that can be computed using exact LP is around N*P=12

In addition to the previous simplification, it is assumed that if a stock price is at the upper (lower) limit, it does not decrease (increase) unless it recombines with an existing node. This simplification makes sure that no new nodes are being created due to the stock limit, reducing the number of nodes by almost one half. One final simplification is by using $U = u^p$ and $L = d^p$ (*p* is an integer) where *u* and *d* represent the step size, and *U* and *L* represent the upper and lower daily stock change limit respectively. As shown in the previous section, randomly choosing *U* will cause the option price to be highly variable. Moreover, this simplification reduces the number of nodes to be calculated and does not limit the flexibility of the model. The number of periods in a day (*P*) can always be increased which would reduce *u* and this in turn would allow us to choose *U* more freely. The model just described will be referred to as **L8**. In order to assess the accuracy of the approximate model, Table 4 compares the approximate and exact LP when the stock price limit is introduced.

Table 4: Exact vs. Approximate pricing of Call option with Stock Price Limit (L=1/U, S=100, K=100, σ=0.1, D=2, P=4, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$, θ=0%)

Daily Stock Limit	Exact LP	Approximate LP
U=1.011	1.09	1.09
U=1.022	1.19	1.31
U=1.033	1.20	1.21
U=1.045	1.20	1.19
U=1.056	1.19	1.19

As shown in the table, the results are similar when the limits are wide. This does not limit the usefulness of the model since in practical applications the stock price limit is always wide (*U*>1.03). Moreover, using the approximate LP would provide a dimensionality reduction. When moving within a day, the number of variables and constrains increase quadratically rather than exponentially. This gives more flexibility to increase the number of periods in a day. The number of variables and constrains do, however, increase exponentially from one day to the next, due to the additional state variable. Nevertheless, in the approximate case the number of days that can be run is observably larger.

5.2.2. Practical Application

Even with the approximation applied in **L8**, the model still cannot handle pricing a one month option since the constraints and variables increase exponentially after each day. However, we can study the one-month option price sensitivity by introducing daily stock price limit in the first couple of days only. Therefore, instead of applying the stock price limit for all the time periods, it is applied to the first *l* days and, afterwards, the case with no stock price limit is applied (**L4)** for the rest of the periods. This hybrid approach will be referred to as **L9**. Table 5 shows some results from this analysis. The results from panel A show that daily stock price limit has a small effect in increasing the option price when the stock volatility is low. Moreover, when daily stock price limit is applied to the first couple of days, it provides a good approximation to the price when the daily limit is applied to all the days; especially when you keep in mind that days further in the future has less value when interest rate is introduced. However, when the stock's volatility is large, the daily stock limit has a strong influence on the option's price. As shown in Panel B, the same daily stock price limit increases the option's price by almost 10% from the no limit case.

Table 5: Price of Put Option (S=100, K=100, U=1.045, P=4, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$, θ=0%, δ=0%)

Panel A: σ=0.1

		Total Number of Days (D)						
		D=2	*D*=3	*D*=4	*D*=5	*D*=10	*D*=15	*D*=21
Stock Limit applied to first *l* days	*l*=2	1.20	1.49	1.73	1.94	2.75	3.37	3.98
	l=3	-	1.49	1.74	1.94	2.75	3.37	3.99
	l=4	-	-	1.74	1.95	2.76	3.37	3.99
	l=5	-	-	-	1.95	f	f	f
	No limit (approximate)	1.19	1.48	1.71	1.92	2.74	3.36	3.98

[f] Computationally infeasible due to the dimensionality of the variables and constraints

Panel B: σ=0.2

		Total Number of Days (D)						
		D=2	*D*=3	*D*=4	*D*=5	*D*=10	*D*=15	*D*=21
Stock Limit applied to first *l* days	*l*=2	2.62	3.28	3.67	4.04	5.59	6.80	8.02
	l=3	-	3.21	3.80	4.15	5.65	6.84	8.05
	l=4	-	-	3.72	4.24	5.70	6.89	8.09
	l=5	-	-	-	4.17	5.76	6.93	8.13
	No limit (approximate)	2.39	2.95	3.43	3.84	5.47	6.71	7.94

[f] Computationally infeasible due to the dimensionality of the variables and constraints

5.3. T+1 Rule

The *T+1* rule is a policy at which the earliest allowed selling date of a stock is one day after the purchase date. This policy is currently implemented in the Chinese stock market. In order to incorporate this restriction in the basic model, the number of days (*D*) and the number of periods in a day (*P*) must be specified. The following constraints can be incorporated in **L3** to apply this rule

(L10)
$$y(n,j) \leq 0 \; \forall j \in \mathcal{F}_n, n = 1,2,\dots,P \qquad (22)$$
$$y(tP+1, j_{tP+1}) + y(tP+2, j_2) + \cdots + y(tP+P, j_P) \leq \alpha(tP, i) \qquad (23)$$
$$\forall\, i \in \mathcal{F}_{tP}, \forall j_{tP+1} \subset i, \forall j_{tP+2} \subset j_{tP+1}, \dots, \forall j_{tP+P} \subset j_{tP+(P-1)}, t = 1,2,\dots,D-1$$

where constraint (22) makes sure no stock can sold in the first day, since the initial stock position is bought at the beginning of the first day. Moreover, in order to make sure that you cannot sell any stock you purchase in any given day after the initial, constraint (23) is applied. Due to the constraint's path dependent nature, approximate LP cannot be utilized; hence, the amount of periods that can be run is limited. Notice that in this problem, the number of periods (*N*) is obtained by multiplying the number of days and number of periods per day. Hence, even if two formulations contain the same number of steps (N_1=N_2), they might produce different results ($D_1 \neq D_2$ and $P_1 \neq P_2$) as illustrated by Table 6. Moreover, as the number of periods in a day is increased, the price of the option increases; this is intuitive since as the number of periods increase, the amount of stocks that can be sold in each step becomes more limited (the pool must be shared among the whole period). Also, when the number of days is increased, the price of the option increases. This behavior can be explained by the increase in the volatility as the number of days is increased.

Table 6: Price of a Call Option with the T+1 rule (S=100, K=100, σ =0.1, Δ=T/N, $u = e^{\sigma\sqrt{\Delta}}$)

		# of steps in a day (P)				
		2	**3**	**4**	**5**	**6**
# of Days (D)	**2**	1.54	1.64	1.73	1.83	1.91
	3	1.74	1.90	1.93	f	f
	4	1.94	2.03	f	f	f
	5	2.12	f	f	f	f
	6	2.28	f	f	f	f

[f] Computationally infeasible due to exponential growth of the model

Similar to the previous section, an approximate LP with a state variable introduced at each stock position at the end of the day can be applied to improve the number of periods at which the option can be priced. However, this is not implemented here since many investors have found a way around it making it practically insignificant. Some investors would borrow a large amount of stock, which would allow them to buy and sell stocks freely afterwards. Other investors buy a large amount of stock initially and hedge them using futures. Again, this allows them to buy and sell stocks freely without worrying about the T+1 rule.

6. Conclusion

The model presented by Edirisinghe et al. (1993) [14] provides a method to price derivatives. However, the model they present is general and does not take into account many practical restrictions. This paper incorporates a rich set of restrictions that are applicable to a wide variety of markets. First, costly shorting is introduced in the model and was shown to affect the put option price. Increasing the shorting cost will eventually lead to the case where shorting is disallowed. Second, the daily stock price limit model is presented along with an approximation that is useful for practical purposes. The impact of the daily stock price limit is shown to be more significant for stock with high volatility. Finally, the *T+1* rule is implemented along with some basic intuition on the results.

Even though the examples presented in this paper are for call and put options, the models are valid for any derivative with a known payoff (*e.g.* futures, currency exchange options). Moreover, the model presented here can be used for nonlinear payoffs such as digital options. This convenience is attributed to the super-replicating criterion used in the basic model.

This paper examines the pricing of derivatives with payoff's dates fixed ahead of time. The pricing of options with payoff on uncertain dates remains an open issue for further research. Moreover, some of the restrictions, including the daily stock price limit, become computationally infeasible within a specific period of time. Devising a dynamic programming model that would allow computing option prices for longer periods of time is also subject to further research.

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